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Block Designs Robust against the Presence of an Aberration in Models with Random Block Effects

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SUMMARY

In this article the problem of finding designs insensitive to the presence of an outlier in a block design for estimating a complete set of orthonormal treatment contrasts has been considered when the block effects are random. Also for a treatment-control block design robustness has been studied for the estimation of the set of elementary contrasts between the effects of each test treatment and a control treatment under the same as sumption on the block effects. The criterion of robustness, suggested by Mandal (1989) in the block design setup for estimating a full set of orthonormal treatment contrasts, is adapted here. It is shown that a randomized block design (RBD) in complete blocks, a balanced incomplete block design (BIBD) and a partially balanced incomplete block design (PBIBD), under certain conditions, in incomplete blocks are robust in the above sense. In the treatment-control setup, a balanced treatment incomplete block design (BTIBD) and a partially balanced treatment incomplete block design (PBTIBD), under certain conditions, are also proved to be robust in the above sense.

Keywords: Robust designs, PBIBD, BTIBD, PBTIBD, Outlier, Mixed effects model.

1. INTRODUCTION

Consider the mixed effects model $\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \mathbf{e}$, in the standard notations, where $\mathbf{Y}$ is the $n\times1$ vector of observations, $\mathbf{X}$ is the $n\times p$ matrix of independent variables with rank s $(\leq p)$, $\boldsymbol{\beta}$ is the $p\times1$ vector of unknown parameters with at least one fixed and at least one random component and $\mathbf{e}$ is the $n\times1$ vector of random errors. Suppose an observation has added to it an 'aberration' c, of unknown magnitude, making it an outlier. It is, however, not known to which observation the aberration is added. The problem is to find designs insensitive to the presence of such an outlier. Box and Draper (1975) were the first to investigate, in this context, the problem of predicting the observed response vector in a response surface model. Later Gopalan and Dey (1976), Mandal (1989), Mandal and Shah (1993), Biswas (2012), Biswas *et. al*. (2013) and some others studied robustness of designs under different contexts in a block design setup. But all these studies were done considering the situation when the parameters are fixed.

In the present paper an attempt has been made to study the robustness of block designs for the estimation of a complete set of orthonormal treatment contrasts against the presence of an outlier when the block effects are random. Also for a treatment-control block design, robustness has been studied for the estimation of the set of elementary contrasts between the effects of each test treatment and a control treatment under the same assumption on the block effects.

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A treatment control design is a very popular and important design in industrial and agricultural trials (cf. Becchofer and Tamhane 1981, Hedayat *et al*. 1988. The designs discussed here are important in the sense that the BLUEs of the treatment contrasts under consideration are affected uniformly, though the wild observation may occur in any one of the responses.

The criterion of robustness, used in this paper, was introduced in Mandal (1989) and later on used by Mandal and Shah (1993) and Biswas (2012). For ready reference a brief description is given here. It should be noted that only connected block designs are considered so that the vector of treatment contrasts of interest, ψ^*,is estimable. For a given design, let$\hat{\psi}^* = \mathbf{H}^*\mathbf{Y}$ be the least squares estimate of ψ^* with dispersion matrix $D(\hat{\psi}^*) = \mathbf{V}^{*-1}$, where $\mathbf{H}^*$ and $\mathbf{V}^*$ are some suitable matrices depending on the model and the design, explicit expressions of which are given in the later sections. If now the u^{th} observation has added to it an aberration c, the change in the i^{th} estimated value $\hat{\psi}_i^*$is given by $\delta_{iu} = ch_{iu}$, where h_{iu} is the $(i,u)^{th}$ element of $\mathbf{H}^*$. Let us write the vector of changes in the estimated value $\hat{\psi}_i^*$ of ψ^* as $\boldsymbol{\delta}_u = (\delta_{1u}, \delta_2 u, \ldots, \delta_{v>u})'$, the value of v' may be v-1 as in a block design or v as in a treatment control design, v being the number of treatments or test treatments as the case may be. A measure of overall discrepancy caused by the effect of c on the u^{th}observation is taken as

$$d_u = \delta_u' \mathbf{V}^* \delta_u \tag{1.1}$$

and the average discrepancy is given by$\bar{d} = \sum_{u=1}^{n} \frac{d_u}{n}$. For the present setups in the next sections, $\bar{d}$is shown to be a constant, independent of the design. Then to choose a design so that at no design point can the addition of c induce large discrepancy in$\hat{\psi}^*$, $d_u$sshould be made as uniform as possible. One convenient measure of this uniformity (cf. Box and Draper, 1975)) is

$$\mathbf{V}(d) = \sum_{u=1}^{n} \frac{(d_u - \bar{d})^2}{n} \tag{1.2}$$

Hence, robustness may be defined as follows.

Definition 1.1: *A design is said to be robust against the presence of an aberration for estimating ψusing a mixed effects model if* $\mathbf{V}(d)$ *is minimum.*

It can be seen that $d_u$sare the diagonal elements of $c^2\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^*$. Hence, minimization of $\mathbf{V}(d)$ implies that d_us should be made as uniform as possible. Therefore, a design is said to be robust if the diagonal elements of $c^2\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^*$ are as uniform as possible.

The paper is organized as follows: the conditions of robustness for block designs and the robustness of a randomized block design (RBD), a balanced incomplete block design (BIBD) and a partially balanced incomplete block design (PBIBD) are derived in Section 2, the conditions of robustness for treatment-control designs and the robustness of a balanced treatment incomplete block design (BTIBD) and a partially balanced treatment incomplete block design (PBTIBD) are derived in Section 3 and finally the discussions on our findings and importance of the results are put forward in Section 4.

2. ROBUST BLOCK DESIGNS WITH RANDOM BLOCK EFFECTS

In the block design setup, let there be b blocks with size k each and v treatments 1, 2,..., v. Let $\mathbf{D}_1^{v\times n}$, $\mathbf{D}_2^{v\times n}$ be the incidence matrices of treatments *vs.* observations and blocks vs. observations respectively and $\mathbf{N}^{v\times b} = \mathbf{D}_1\mathbf{D}_2'$ be the corresponding treatments vs. blocks incidence matrix with elements n_{ij}, denoting the number of times the i^{th} treatment occurs in the j^{th} block, i = 1,..., v and j = 1, 2,..., b. The incidence matrices and their functions play an important role in characterizing a robust design. **I**, **J** and **1** denote respectively the identity matrix, the matrix with all elements 1 and the sum vector of suitable orders. Let us restrict to connected designs.

Consider the mixed effects additive model

$$\boldsymbol{Y} = \mu\mathbf{1} + \mathbf{D}_1'\boldsymbol{\tau} + \mathbf{D}_2'\boldsymbol{\beta} + \mathbf{e} \tag{2.1}$$

where μ is an additive constant, $\tau = (\tau_1,..., \tau_v)'$ is the vector of unknown treatment effects and $e = (e_1,e_2,...,e_n)$ is the random error vector with $E(\boldsymbol{e}) = \mathbf{0}$ and $D(\boldsymbol{e}) = \sigma^2 I$. Also let $\boldsymbol{\beta} = (\beta_1, \beta_2, \cdots, \beta_b)$ be the random vector of block effects with E(β)= 0 and D (β) $=\sigma_1^2\mathbf{I}$. The block effects and errors are assumed to be independent. So from (2.1) it follows that

$$E(\mathbf{Y}) = \mu\mathbf{1} + \mathbf{D}_1'\boldsymbol{\tau} \quad (2.2)$$

$$D(\mathbf{Y}) = \sigma^2 I + \sigma_1^2\mathbf{D}_2'\mathbf{D}_2 = diag(K, K,..., K) = \Sigma \quad (2.3)$$

where

$$K = \sigma^2\mathbf{I}_b + \sigma_1^2\mathbf{J}_b. \quad (2.4)$$

Now,

$$\Sigma^{-1} = diag(K^{-1}, K^{-1}, \cdots, K^{-1}) \quad (2.5)$$

It can be easily seen (cf. Bose (1975)) that

and

$$\mathbf{K} = w\mathbf{I} - \frac{w-\tilde{w}}{k}\mathbf{J} \quad (2.6)$$

$$\Sigma^{-1} = w\mathbf{I} - \frac{w-\tilde{w}}{k}\mathbf{D}_2'\mathbf{D}_2, \quad (2.7)$$

where

$$w = \frac{1}{\sigma^2}, \tilde{w} = \frac{1}{\sigma^2+k\sigma_1^2} \quad (2.8)$$

Now, with $C^* = ((c_{ij}^*))$ as the information matrix for the mixed effects model, the reduced normal equations are given as (cf. Bose (1975))

$$\mathbf{C}^*\hat{\tau}^* = \mathbf{Q}^* \quad (2.9)$$

where $\hat{\tau}^* = \mathbf{C}^{*-1}\mathbf{Q}^*$ is a solution to the reduced normal equations, $\mathbf{C}^{*-1} = ((c_{ij}^*))$ being a g-inverse of $\mathbf{C}^*$. Without loss of generality, $\bar{\mathbf{C}}^*$ is chosen to be the Moore-Penrose generalized inverse. Moreover, with $\mathbf{R} = diag(r_1,...,r_v)$, where r_i is the number of times the i^{th} treatment is replicated in the whole design, $i = 1,...,v$ and $\mathbf{r} = (r_1,...,r_v)$, (cf. Bose (1975)),

$$\mathbf{C}^* = w\mathbf{C} + \tilde{w}\tilde{\mathbf{C}}, \mathbf{Q}^* = w\mathbf{Q} + \tilde{w}\tilde{\mathbf{Q}}, \quad (2.10)$$

where

$$\mathbf{C} = \mathbf{R} - \frac{1}{k}\mathbf{N}\mathbf{N}', \tilde{\mathbf{C}} = \frac{1}{k}\mathbf{N}\mathbf{N}' - \frac{1}{n}\mathbf{r}\mathbf{r}' \quad (2.11)$$

$$\mathbf{Q} = \mathbf{G}\mathbf{Y}, \tilde{\mathbf{Q}} = \tilde{\mathbf{G}}\mathbf{Y} \quad (2.12)$$

Here

$$\mathbf{G} = \mathbf{D}_1 - \frac{1}{k}\mathbf{N}\mathbf{D}_2, \tilde{\mathbf{G}} = \mathbf{D}_1 - \mathbf{G}\frac{1}{n}r_1, \quad (2.13)$$

such that

$$\mathbf{Q}^* = \mathbf{G}^*\boldsymbol{Y}, \quad (2.14)$$

where

$$\mathbf{G}^* = w\mathbf{G} + \tilde{w}\tilde{\mathbf{G}}. \quad (2.15)$$

From the above equations, it easily follows that

$$\mathbf{C}^*1 = 0 \quad (2.16)$$

$$\mathbf{1}'\mathbf{G}^* = \mathbf{0}', \mathbf{G}^{*\prime}\mathbf{G}^* = \mathbf{C}^* \quad (2.17)$$

Let us restrict to the class of equireplicated designs with a common replication number *r*.

The complete set of orthonormal vector of treatment contrasts under consideration can be taken as

$$\psi = \mathbf{P}\tau \quad (2.18)$$

where **P** is a $(v - 1) \times v$ matrix such that

$\tilde{\mathbf{O}} = \left(\frac{1}{\sqrt{v}}1, \mathbf{P}'\right)$ is an orthogonal matrix of order *v*. It is clear that P satisfies

$$\mathbf{P1} = \mathbf{0}, PP = I, \mathbf{P}'\mathbf{P} = \mathbf{I} - \frac{1}{v}\mathbf{J}. \quad (2.19)$$

Then it follows from (2.9), (2.14), (2.18) and (2.19) (cf.Mandal (1989)) that, for a given design, the least squares estimate of $\boldsymbol{\psi}$ is

$$\hat{\boldsymbol{\psi}}^* = \mathbf{P}\hat{\tau}^* = (\mathbf{P}\mathbf{C}^*\mathbf{P}')^{-1}\mathbf{P}\mathbf{Q}^* = \mathbf{H}^*\mathbf{Y} \quad (2.20)$$

where

$$\mathbf{H}^* = (\mathbf{P}\mathbf{C}^*\mathbf{P}')^{-1}\mathbf{P}\mathbf{G}. \quad (2.21)$$

It is easy to see that $E(\hat{\psi}^*) = \psi$ and $\mathbf{D}(\hat{\psi}^*) = (\mathbf{P}\mathbf{C}^*\mathbf{P}')^{-1} = \boldsymbol{V}^{*-1}$ (say), since $D(\mathbf{Q}^*) = \mathbf{C}^*$, (cf. Bose (1975)). Since $\mathbf{P}\tau$ is estimable, it is easy to see using (2.21) that

$$H^*H^{*\prime} = (\mathbf{PC}^*\mathbf{P}')^{-1} = \boldsymbol{V}^{*-1}. \quad (2.22)$$

Now, it can be easily shown that

$$\bar{d} = \frac{1}{n}\sum_{u=1}^{n} d_u = \frac{c^2(v-1)}{n} \quad (2.23)$$

which is a constant independent of the design.

As mentioned earlier, a design for which the diagonal elements of $c^2\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^*$are as uniform as possible is robust, we need to work out the structure of $\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^*$.

$$\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^* = \mathbf{G}^{*\prime}\mathbf{P}'(\mathbf{PC}^*\mathbf{P}')^{-1}\mathbf{V}^*(\mathbf{PC}^*\mathbf{P}')^{-1}\mathbf{PG}^*$$
$$= \mathbf{G}^{*\prime}\mathbf{P}'(\mathbf{PC}^*\mathbf{P}')^{-1}\mathbf{PG}^*$$
$$= \mathbf{G}^{*\prime}\mathbf{P}'(\mathbf{PC}^*\mathbf{P}')$$
$$= \mathbf{G}^{*\prime}\mathbf{C}^{*\prime}\mathbf{G}^* \quad (2.24)$$

since $\mathbf{D}(\hat{\psi}^*) = (\mathbf{PC}^{*-1}\mathbf{P}')$

Using (2.13) and (2.15) and since $\bar{\mathbf{C}}^*$is a Moore-Penrose g-inverse of $\mathbf{C}^*$ so that$\bar{\mathbf{C}}^*1=\mathbf{0}$,

$$\mathbf{G}^{*\prime}\bar{\mathbf{C}}^*\mathbf{G}^* = (w\mathbf{G}' + \tilde{w}\tilde{\mathbf{G}}')\bar{\mathbf{C}}^*(w\mathbf{G}' + \tilde{w}\tilde{\mathbf{G}})$$
$$= w^2\mathbf{G}'\bar{\mathbf{C}}^*\mathbf{G} + \mathrm{w}\tilde{w}(\mathbf{G}'\bar{\mathbf{C}}^*\tilde{\mathbf{G}} + \tilde{\mathbf{G}}'\bar{\mathbf{C}}^*\mathbf{G}) + w^2\mathbf{G}'\bar{\mathbf{C}}^*\tilde{\mathbf{G}}$$
$$= (w^2 + w\tilde{w} + \tilde{w}^2)\mathbf{G}'\bar{\mathbf{C}}^*\tilde{\mathbf{G}} + (w\tilde{w} - \tilde{w}^2)\mathbf{G}'\bar{\mathbf{C}}^*\mathbf{D}_1 + \frac{\mathbf{1}}{\boldsymbol{n}}(w^2 - w\tilde{w})\mathbf{G}'\bar{\mathbf{C}}^*\mathbf{J} + (w\tilde{w} + \tilde{w}^2)\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{G} + \frac{\boldsymbol{r}}{\boldsymbol{n}}(\tilde{w}^2 - w\tilde{w})\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{G} + \tilde{w}^2\left(\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{D_1} + \frac{\mathbf{r^2}}{\mathbf{n^2}}\mathbf{J}\bar{\mathbf{C}}^*\mathbf{J}\right)$$

$$\mathbf{N} = \frac{r}{n}\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{J} - \frac{r}{n}\mathbf{J}\bar{\mathbf{C}}^*\mathbf{D}_1$$

$$= (w - \tilde{w})^2\mathbf{G}'\bar{\mathbf{C}}^*\mathbf{G} + (w\tilde{w} - \tilde{w}^2)(\mathbf{G}'\bar{\mathbf{C}}^*\mathbf{D}_1 + \mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{G}) + \tilde{w}^2\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{D}_1$$

$$= w^2\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{D}_1 + \frac{(w\tilde{w}-\tilde{w}^2)}{k}\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{D}_2 + \frac{(w\tilde{w}-\tilde{w}^2)}{k}\mathbf{D}_2'\mathbf{N}\bar{\mathbf{C}}^*\mathbf{D}_1 + \frac{(w-\tilde{w})^2}{k^2}\mathbf{D}_2'\mathbf{N}\bar{\mathbf{C}}^*\mathbf{ND}_2 \quad (2.25)$$

Thus, from (2.25) a set of suffcient conditions for a design to be robust is that the diagonal elements of each of the four terms in the expression are separately as uniform as possible. Hence the following theorem can be stated.

Theorem 2.1: In the class of all connected proper block designs, a design is robust against the presence of an aberration for estimating *ip* in the mixed model setup if the diagonal elements of $\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{D}_1$, $\mathbf{D}_1'\bar{\mathbf{C}}^*\mathbf{ND}_2$and that of $\mathbf{D}_2'\mathbf{N}\bar{\mathbf{C}}^*\mathbf{ND}_2$ are separately as uniform as possible.

For a design to be robust, the conditions in the above theorem are sufficient. This requires

$$\mathbf{e}_u'\bar{\mathbf{C}}^*\mathbf{e}_u = \textit{constant},\ \mathbf{e}_u'\bar{\mathbf{C}}^*\boldsymbol{n}_u^* = = \textit{constant},\ \mathbf{n}_u^*\bar{\mathbf{C}}^*\boldsymbol{n}_u^* = = \textit{constant}\ \forall u, \quad (2.26)$$

where the u^{th} observation occurs in the *j*th block with the i^{th} treatment. Here, $\mathbf{e}_u$and $\boldsymbol{n}_u^*$are the u^{th}columns of $\mathbf{D}_1$ and $\mathbf{ND}_2$ respectively. From Theorem 2.1 and (2.26), a set of sufficient conditions for a block design to be robust can be obtained and is given below.

i) c^{*ii} is constant for all $i = 1,2,...,v$,

ii) $\bar{\mathbf{C}}^*_{\boldsymbol{n}j}$= g_j, j = 1, 2,..., b,

where g_j is a v dimensional vector with two distinct elements, independent of j, corresponding to the elements 1 and 0 in n_j, j = 1, 2,..., b.

From the above discussion the following observations may be drawn.

Corollary 2.1: For an RBD and a BIBD the two conditions as stated before are satisfied and hence they are robust.

Example 2.1: Let us consider a BIBD from Raghavarao (1971) with parameters v = 13, r = 4, k = 4, b = 13, λ = 1. The blocks are (0,1, 3, 9), (3, 4, 6, 12), (6, 7, 9, 2), (9, 10, 12, 5), (12, 0, 2, 8), (1, 2, 4, 10), (4, 5, 7, 0), (7, 8, 10, 3), (10, 11, 0, 6), (2, 3, 5, 11), (5, 6, 8, 1), (8, 9, 11, 4), (11, 12, 1, 7) numbered as 1, 2,..., 13 respectively. It follows from our computation that, c^{*w} = 0.2833762 for all i = 1,2,...,13 and $\bar{\mathbf{C}}^*_{\boldsymbol{n}j}$, is a column vector with elements 0.212532 in those positions where 1 occurs in $\mathbf{n}_j$ and -0.0944587 where 0 occurs in $\mathbf{n}_j$, for all j = 1, 2,..., 13. For this computation σ is assumed to be 1 and σ_1 is assumed to be 5 to keep a distinction between the two variances. Elaborately, when j= 1, g_1= (0.212532, 0.212532, –0.0944587, 0.212532, –0.0944587, –0.0944587, –0.0944587, –0.0944587, –0.0944587, –0.0944587, –0.0944587, –0.0944587)'. Here, the element 0.212532 occurs for treatment positions 0,1,3,9 i.e. those treatments present in the 1st block and –0.0944587 occurs for treatment positions 2,4,5,6,7,8,10,11,12 i.e. those not present in block 1. Hence, this design is robust as it satisfies the above two stated conditions.

In the PBIB design setup, the two conditions stated above lead to the following corollary.

Corollary 2.2: A PBIB design with two-associate classes is robust against the presence of an aberration for estimating *ip* in the mixed model setup if for it any treatment *i* appearing in block *j*, the number of first associates of *i* occurring in the block is a constant independent of *i* and *j*, *i* = 1,2,..., *v*,*j*= 1, 2,..., *b*.

Examples of such PBIBDs are cited in Gopalan and Dey (1976). One such example is discussed here elaborately.

Example 2.2: Let us consider a two-associate semi-regular group divisible PBIBD from *Clatworthy* (1973), *SR4* with parameters v = 4, r = 4, k = 2, b = 8, m = 2, n = 2, λ1 = 0, λ2 = 2. The groups of the association scheme are (1, 3) and (2, 4). The blocks are (1, 2), (3, 4), (4, 1), (2, 3), (1, 2), (3, 4), (4, 1), (2, 3) numbered as 1, 2,..., 8 respectively. Obviously, this design satisfies the condition of Corollary 2.1. Now, we show that it satisfies the conditions of Theorem 2.1. It follows from our computation that $\mathbf{C}^{*ii}$= 0.3076923 for all i = 1, 2, 3,4 and $\bar{\mathbf{C}}^*_{\boldsymbol{n}j}$, is a column vector with elements 0.245192 in those positions where 1 occurs in n_j and -0.245192 where 0 occurs in n_j, j = 1,2,...,8. For this computation σ is assumed to be 1 and σ_1 is assumed to be 5 to keep a distinction between the two variances. Elaborately, whenj = 1, g_1 = (0.245192, 0.245192, -0.245192, -0.245192)'. Here, the element 0.245192 occurs for treatment positions 1, 2 i.e. those treatments present in the 1st block and –0.245192 occurs for treatment positions 3, 4 i.e. those not present in block 1. Hence, this design is robust as it satisfies the above two stated conditions.

3. ROBUST TREATMENT-CONTROL BLOCK DESIGNS WITH RANDOM BLOCK EFFECTS

In this section robustness of treatment-control designs is considered. The formulation of the criterion of robustness remains the same as in a block design setup, except that here $(v + 1)$ treatments are involved with one control treatment 0, with replication r_0 and v test treatments with a common replication number r. Let us restrict to connected designs.

The information matrix for a single control and v test treatments can be partitioned as

$$\mathrm{C}^* = \begin{bmatrix} c_{11} & c_1' \\ c_1 & c_1^* \end{bmatrix} \tag{3.1}$$

$\boldsymbol{c}_{11}$ is the element in the first row and column of $\mathbf{C}^*$, $\boldsymbol{c}_1'$ = (c_{12}, c_{13},...,c'_{1v+1}) and $\boldsymbol{c}_1^*$ = (($\boldsymbol{c}_{ij}^*$)), i, j = 2, 3,..., $\overline{v+1}$. As the design is connected, a g-inverse of $\mathbf{C}^*$ is taken as

$$\bar{\mathrm{C}}^* = \begin{bmatrix} 0 & 0' \\ 0 & c_1^{*-1} \end{bmatrix} \tag{3.2}$$

without loss of generality.

The set of elementary contrasts between the effects of each test treatment and the control treatment is given by

$$\boldsymbol{\psi} = \mathbf{P}\tau, \tag{3.3}$$

where P = (-1,I) is a $v \times (v + 1)$ matrix satisfying

$$\mathrm{P}1 = 0, \mathrm{P}'1 = (-v, 1'),$$

$$\mathrm{P}'\,\mathrm{JP} = \begin{bmatrix} v^2 & -v1' \\ -v1 & \mathrm{J} \end{bmatrix}$$

$$\mathrm{P}'\mathrm{P} = \begin{bmatrix} v & -1' \\ -1 & \mathrm{I} \end{bmatrix} \tag{3.4}$$

Then it follows from (2.14) and (3.3) that, for a given design, the least squares estimate of *ip* is

$$\widehat{\boldsymbol{\psi}}^* = \mathbf{P}\hat{\boldsymbol{\tau}}^* = \mathbf{H}^*\mathbf{Y}, \mathbf{H}^* = \mathbf{P}\bar{\mathbf{C}}^*\mathbf{G}^* \quad (3.5)$$

It is easy to see that $E(\widehat{\boldsymbol{\psi}}^*) = \boldsymbol{\psi}$and $D(\widehat{\boldsymbol{\psi}}^*) = \mathbf{P}\bar{\mathbf{C}}^*\mathbf{P}' = \mathbf{V}^{*-1}$ (say), assuming $\bar{\mathbf{C}}^*$to be a symmetric g-inverse of C. Since $\mathbf{P}\boldsymbol{\tau}$is estimable, it is easy to see that using (3.5)

$$\mathbf{H}^*\mathbf{H}^{*\prime} = \mathbf{P}\bar{\mathbf{C}}^*\mathbf{P}' = \boldsymbol{V}^{*-1} \quad (3.6)$$

Now, as before it can be easily shown for the treatment-control setup as well that

$$\bar{d} = \frac{1}{n}\sum_{u=1}^{n} d_u = \frac{vc^2}{n} \quad (3.7)$$

which is a constant independent of the design.

Using (3.5) and (3.6)

$$\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^* = (\mathbf{G}^{*\prime}\bar{\mathbf{C}}^*\mathbf{P}')\mathbf{V}(\mathbf{P}\bar{\mathbf{C}}^*\mathbf{G}^*)$$

$$= \mathbf{G}^{*\prime}\bar{\mathbf{C}}^*\mathbf{P}'(\mathbf{P}\bar{\mathbf{C}}^*\mathbf{P}')^{-1}(\mathbf{P}\bar{\mathbf{C}}^*\mathbf{G}^*) \quad (3.8)$$

Now, using (3.2) in (3.8) and after a brief algebra, it follows that

$$\mathbf{H}^{*\prime}\mathbf{V}^*\mathbf{H}^* = \boldsymbol{G}_1^{*\prime}\boldsymbol{C}_1^{*-1}\boldsymbol{G}_1^* \quad (3.9)$$

Here, $\mathbf{G}^*$ is partitioned as

$$\mathbf{G}^* = \begin{bmatrix} g_{\mathbf{0}}^{*\prime} \\ \boldsymbol{G}_1^* \end{bmatrix} =^0, \quad (3.9)$$

where $g_{\mathbf{0}}^{*\prime}$ is the first row of the matrix $\mathbf{G}^*$ corresponding to the control treatment and $\boldsymbol{G}_1^*$ is the sub-matrix of $\mathbf{G}^*$ corresponding to the *v* test treatments. Because of (2.19), $g_{\mathbf{0}}^*$ and $\boldsymbol{G}_1^*$ are related through

$$1'\boldsymbol{G}_1^* = -g_{\mathbf{0}}^* \quad (3.10)$$

Let us restrict to the class of designs for which the information matrix $\boldsymbol{C}_1^{*-1}$ is completely symmetric (c.s) or equivalently $\boldsymbol{C}_1^{*-1}$is c.s with $\boldsymbol{C}_1^{*-1}$= a_1I + a_2J, $a_1$and $a_2$are some constants dependent on the design. Because of (3.10), (3.9) can be expressed as

$$\mathbf{H}'\mathbf{V}\mathbf{H} = a_1\mathbf{G}_1^{*\prime}\mathbf{G}_1^* + a_2\mathrm{g}_0^*\mathrm{g}_0^{*\prime} \quad (3.11)$$

Hence, the following theorem can be obtained.

Theorem 3.1: In the class of all connected treatment-control block designs, a design is robust against the presence of an aberration for estimating ψif

i) $\mathbf{C}_1^*$ is c.s and

ii) After a brief algebra it is derived that

the diagonal elements of $\mathrm{g}_0^*\mathrm{g}_0^{*\prime}$ and that of $\mathbf{G}_1^{*\prime}, \mathbf{G}_1^*$ are separately as uniform as possible.

$$\mathrm{g}_0^{*\prime} = \left[\frac{(\widetilde{w}-w)}{k}\right]\mathbf{D}_2'\mathbf{D}_2 d_0 - \frac{r\widetilde{w}}{n}\mathbf{I} + \mathrm{w}d_0 \quad (3.12)$$

$$\mathbf{G}_1^* = w\mathbf{D}_{11}\left[\frac{(\widetilde{w}-w)}{k}\right]\mathbf{D}_{11}\mathbf{D}_2' - \frac{r\widetilde{w}}{n}\mathbf{J}$$

$$\mathbf{G}_1^* = w\mathbf{D}_{11} + \frac{(\widetilde{w}-w)}{k}\mathbf{D}_{11}\mathbf{D}_2'\mathbf{D}_2\frac{r\widetilde{w}}{n}\mathbf{J} \quad (3.13)$$

where $\mathbf{D}_1 = \begin{bmatrix} d_0' \\ \mathbf{D}_{11} \end{bmatrix}$. Again, from (3.11) and (3.12) it follows that

$$\begin{aligned}
&\mathrm{g}_0^*\mathrm{g}_0^{*\prime} \\
&= w^2 d_0 d_0' \\
&+ w\left[\frac{\widetilde{w}-w}{k}\right](d_0 d_0'\mathbf{D}_2'\mathbf{D}_2 + \mathbf{D}_2'\mathbf{D}_2 d_0 d_0') \\
&- \frac{r_0 w\widetilde{w}}{n}(d_0\mathbf{1}' + \mathbf{1}d_0') \\
&+ \left[\frac{(\widetilde{w}-w)}{k}\right]^{\mathbf{2}}\mathbf{D}_2'\mathbf{D}_2 d_0 d_0'\mathbf{D}_2'\mathbf{D}_2 \\
&- \frac{r_0\widetilde{w}}{n}\frac{\widetilde{w}-w}{k}(\mathbf{D}_2'\mathbf{D}_2 d_0\mathbf{1}' + \mathbf{1}d_0'\mathbf{D}_2'\mathbf{D}_2) \\
&+ \widetilde{w}^2\frac{r_0^2}{n^2}\mathbf{J}
\end{aligned} \quad (3.14)$$

$$\mathbf{G}_1^{*\prime}\mathbf{G}_1^* = w^2\mathbf{D}_{11}'\mathbf{D}_{11} + w\frac{\widetilde{w}-w}{k}(\mathbf{D}_{11}'\mathbf{D}_{11}\mathbf{D}_2'\mathbf{D}_2 - \mathbf{D}_2'\mathbf{D}_2\mathbf{D}_{11}'\mathbf{D}_{11}) - \frac{rw\widetilde{w}}{n}(\mathbf{D}_{11}'\mathbf{J} + \mathbf{J}\mathbf{D}_{11}) + \left[\frac{(\widetilde{w}-w)}{k}\right]^{\mathbf{2}}\mathbf{D}_2'\mathbf{D}_2\mathbf{D}_{11}\mathbf{D}_2'\mathbf{D}_2 - \frac{r\widetilde{w}}{n}\frac{\widetilde{w}-w}{k}\Big(\mathbf{D}_2'\mathbf{D}_2\mathbf{D}_{11}'\mathbf{J} + \mathbf{J}\mathbf{D}_{11}\mathbf{D}_2'\mathbf{D}_2 + \widetilde{w}^2\frac{r_0^2 v}{n^2}\mathbf{J}\Big) \quad (3.15)$$

Let us denote the $(u,u)^{th}$element of a matrix A by $(A)_{u,u}$. So, it can be observed from (3.14) and (3.15) respectively that if the u^{th}unit occurs in the j^{th} block with the i^{th} test treatment

$$g_0^* g_0^{*\prime} = \left[\frac{\tilde{w}-w}{k} r_{0j} - \frac{r_0 \tilde{w}}{n}\right]^2 \quad (3.16a)$$

and if the u^{th} unit occurs in the j^{th} block with the control treatment, where $r_{0j}$is thenumber of times the control treatment occurs in the j^{th} block

$$g_0^* g_0^{*\prime} = w^2 + 2w \frac{\tilde{w}-w}{k} r_{0j} - 2\frac{r_0 \tilde{w}}{n} + \left[\frac{\tilde{w}-w}{k} r_{0j} - \frac{r_0 \tilde{w}}{n}\right]^2 \quad (3.16b)$$

Again, if the u^{th} unit occurs in the j^{th} block with the i^{th} test treatment

$$(\mathbf{G}_1^{*\prime}\, \mathbf{G}_1^*)_{uu} = w^2 + 2w \frac{\tilde{w}-w}{k} n_{ij} - 2\frac{r_0 w \tilde{w}}{n_{uu}} + \left[\frac{\tilde{w}-w}{k} r_{0j} - \frac{r \tilde{w}}{n}\right]^2 \quad (3.17a)$$

and if the u^{th} unit occurs in the j^{th} block with the control treatment

$$(\mathbf{G}_1^{*\prime}\, \mathbf{G}_1^*)_{uu} = w^2 + 2w \frac{\tilde{w}-w}{k} n_{ij} - 2\frac{r_0 w \tilde{w}}{n_{uu}} + \left[\frac{\tilde{w}-w}{k}\right]^2 \sum_{u=1}^{n} {n_{ij}}^2 - 2\frac{r\tilde{w}(\tilde{w}-w)}{n} + \tilde{w}^2 \frac{r^2 v}{n^2} \quad (3.17b)$$

Thus, from (3.16a) and (3.17a), a set of sufficient conditions for a treatment-control design to be robust can be obtained and is given below.

1. $\mathbf{C_1^*}$ is completely symmetric and
2. The frequencies in the non-empty cells for the test treatments are all equal and the number of times the control treatment occurs in each block over the whole design is a constant.

Corollary 3.1: It is clear that for an R-type BTIBD both the above conditions in are satisfied and hence it is robust.

4. CONCLUSION

1. All the results in the context of fixed block effects regarding the robustness of RBD, BIBD, PBIBD in their respective setups (cf. Mandal 1989, Mandal and Shah 1993) are satisfied with random block effects as well.
2. In the treatment-control setup, an R-type BTIBD is proved to be robust in the above sense as was true with fixed block effects in Biswas (2012). Moreover, it is seen following Biswas (2012) that a PBTIBD with two-associate classes and random block effects with a constant number of first associates of any test treatment in any block is robust as well.
3. For S-type BTIB designs, the control treatment does not occur an equal number of times in all the blocks and since the conditions stated in Section 3 are only sufficient for a design to be robust, nothing can be concluded about the robustness of such designs. However, a separable S-type BTIB design (Biswas (2012)) with random block effects can be considered a nearly robust design among competing designs.

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